11/14 Recitation Solutions

1. Let

$$A = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{5} & 0 & 0 & 2/\sqrt{5} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -2/\sqrt{5} & 0 & 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}^T$$

be an SVD.

   (a) What is the rank of A?

   Since there are two nonzero singular values, the rank of A is 2.

   (b) Find bases of the four fundamental subspaces of A.

   The first $r = 2$ columns of V form a basis for $C(A^T)$ and the last column forms a basis for $N(A)$. The first $r = 2$ columns of U form a basis for $C(A)$ and the last two columns form a basis for $N(A^T)$.

   (c) What are the eigenvalues of A^TA?

   The squares of the diagonal entries of Σ are eigenvalues of A^TA. Hence A^TA has eigenvalues $5, 4$ and 0. Note that these are the only eigenvalues since A^TA is 3×3.

   (d) Are the columns of A are linearly independent?

   Since A^TA has an eigenvalue of 0, A^TA is not invertible. It follows that the columns of A are not linearly independent.

2. Prove that $N(A^TA) = N(A)$ as follows.

   (a) Show $N(A) \subseteq N(A^TA)$.

   Let $\mathbf{x} \in N(A)$. Then $A\mathbf{x} = \mathbf{0}$ and so $(A^TA)\mathbf{x} = A^T(A\mathbf{0}) = A^T\mathbf{0} = \mathbf{0}$. Thus $\mathbf{x} \in N(A^TA)$ and so $N(A) \subseteq N(A^TA)$.

   (b) Show $N(A^TA) \subseteq N(A)$ (hint: notice that $\mathbf{x}^T(A^TA)\mathbf{x} = 0$ for all $\mathbf{x} \in N(A^TA)$)

   Let $\mathbf{x} \in N(A^TA)$. Then $A^TA\mathbf{x} = \mathbf{0}$. It follows that:

$$0 = \mathbf{x}^T(A^TA)\mathbf{x} = (\mathbf{x}^TA^T)(A\mathbf{x}) = (A\mathbf{x})^TA\mathbf{x} = (A\mathbf{x}) \cdot (A\mathbf{x}) = ||A\mathbf{x}||^2.$$

Now $||A\mathbf{x}|| = 0$ implies $A\mathbf{x} = \mathbf{0}$; hence $\mathbf{x} \in N(A)$ and so $N(A^TA) \subseteq N(A)$.

(c) Deduce that $N(A^TA) = N(A)$.

Since $N(A^TA) \subseteq N(A)$ and $N(A) \subseteq N(A^TA)$, we have that $N(A) = N(A^TA)$.

3. Complete the two problems in the **Julia13SVD**.

Problem 1: Let $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

(a) Find an SVD of A.

(b) Diagonalize A.

(c) Notice that these are the same! What makes A special here? Play around with other matrices having similar properties as A and try to determine conditions under which SVD and diagonalization coincide. For such a matrix A, explain why the singular values are the same as the eigenvalues.

It turns out that if A is positive semi-definite symmetric matrix, then $A = U\Sigma V^T$ provides a diagonalization of A. Let $\lambda_1 \geq \cdots \geq \lambda_r$ be the nonzero eigenvalues of A^TA. Since $A^T = A$, we have $A^TA = A^2$; hence the nonzero eigenvalues of A^TA are simply the squares of the nonzero eigenvalues of A. Moreover, since A is positive semi-definite, its eigenvalues are non-negative. Thus the nonzero eigenvalues of A are simply $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r}$, which are the singular values of A.